Algebraic Geometry

Lecture Notes

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1 Lecture I

We begin by recalling the theory of affine algebraic varieties, which we glimpsed during last year's lectures and exercises. This is useful both to build intuition and motivation for the language of schemes, and also to remember some fundamental results from commutative algebra. I advise you to look into the numerous references (the most important are listed on the Moodle page). The aim of this first lecture is also to define some of the main objects which we shall later study via the language of schemes.

1.1 Affine algebraic varieties

Let K be an algebraically closed field, which we fix from now on. The weak Nullstellensatz says that we can identify K^n with the set of maximal ideals of $K[x_1, \dots, x_n]$, and it clearly expresses the first fundamental bridge between algebra and geometry. The full Nullstellensatz instead says that for every ideal $I \subset K[x_1, \dots, x_n]$ we have

$$I(V(I)) = \sqrt{I}.$$

(Why does this imply the weak form?)

Recall that we can endow K^n with the Zariski topology, where the closed subsets are of the form $V(I)$ for $I \subset K[x_1, \dots, x_n]$. We also call the subsets $V(I)$ the *algebraic subsets* of K^n. These are very explicitly defined: since $K[x_1, \cdots, x_n]$ is Noetherian, every ideal is finitely generated. Write $I = (f_1, \dots, f_d)$, then

$$V(I) = \{(x_i) \in K^n : \ f_1(x_1, \dots, x_n) = \cdots = f_d(x_1, \dots, x_n) = 0\},$$

which in turn corresponds to maximal ideals $\mathfrak{m} \subset K[x_1, \dots, x_n]$ with $I \subset \mathfrak{m}$.

These subsets are the basic building blocks of algebraic geometry, and we shall spend some time understanding their properties. First of all, we can endow $V(I)$ with a topology in two equivalent ways (verify this): either restrict the Zariski topology of K^n to $V(I)$, or identify $V(I)$ with the set of maximal ideals $\mathfrak{m} \subset A := K[x_1, \dots, x_n]/I$ and endow it with the topology generated by

$$V(J) := \{\mathfrak{m} \subset A : \ J \subset \mathfrak{m}\}$$

for ideals $J \subset A$.

The topology of $V(I)$ is very different from what we are normally used to, and enjoys many finiteness properties:

Definition 1.1. A topological space X is said to be *Noetherian* if any descending chain of closed subsets

$$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$$

eventually stabilizes.

The fact that $K[x_1, \cdots, x_n]$ is a Noetherian ring then readily implies that any $V(I)$ is also Noetherian. Recall that we defined the topological space $\mathrm{Spec}(A)$ for any commutative ring. Show that if A is Noetherian then $\mathrm{Spec}(A)$ is Noetherian (is this an if and only if?). Note, on the other hand, that the usual topology of $\mathbb{R}^n$ is never Noetherian for $n \geq 1$.

Theorem 1.2 (Noetherian induction). *Let X be a Noetherian topological space and P be a property of the closed subsets of X. Assume $P(\emptyset)$ is true, and that for every closed subset $Z \subset X$, the fact that $P(W)$ is true for every proper closed $W \subsetneq Z$ implies that also $P(Z)$ is true. Then P is true for all closed subsets of X.*

This or similar formulations appear many times in proofs, for example when studying coherent cohomology. We use it now to show that every $V(I)$ can be decomposed uniquely into a union of irreducible subsets:

Definition 1.3. A Noetherian topological space X is *irreducible* if for every two closed subsets $X_1, X_2 \subsetneq X$ we have $X_1 \cup X_2 \subsetneq X$.

Recall that $V(I)$ is irreducible if and only if $\sqrt{I}$ is prime.

Theorem 1.4. *Every closed subset $V(I) \subset K^n$ can be decomposed uniquely as a finite union of distinct irreducible closed subsets.*

Proof. Prove it using Noetherian induction. Compare it to the proof that every radical ideal of $K[x_1, \dots, x_n]$ decomposes uniquely into a product of prime ideals. We omit the details. ☐

The subsets $V(I)$ with I prime are called *affine algebraic varieties.*

Remark 1.5. To gain geometric intuition for algebraic varieties, assume that $\mathrm{char}(K) = 0$ and that $|K| \leq |\mathbb{C}|$ (this assumption is very reasonable). Then one can embed $K \subset \mathbb{C}$ and consider $V(I)$ as an algebraic subset of $\mathbb{C}^n$. But this is also closed for the usual topology, and we can study its properties with tools from topology, real analysis, and complex analysis. As one will see, many topological properties of $V(I)$ can be obtained by purely algebraic methods without leaving the language of algebraic geometry.

Many times we can also embed $K \subset \mathbb{R}$, and $V(I)$ yields a closed subset of $\mathbb{R}^n$. Although this can still be used to get geometric insights into $V(I)$, we remark that $V(I)$ may behave strangely (e.g. it may be empty). This is why classical algebraic geometry is formulated over algebraically closed fields: so that we see all points of $V(I)$ over K.

The topology of irreducible Noetherian topological spaces is very coarse:

Proposition 1.6. *If X is an irreducible topological space and $U \subset X$ is a non-empty open subset, then U is dense.*

Proof. Assume that $\overline{U} \neq X$, so $V = X \setminus \overline{U}$ is open and non-empty. Note that $V \cap U = \emptyset$ and that $X = \overline{U} \cup \overline{V}$, contradicting irreducibility. ☐

Thus affine algebraic varieties are never Hausdorff (T_2) in the classical sense. In fact, the right concept is the one of *separatedness*, as we shall see later in the course.

Observation 1.7. *If we wish to have a faithful dictionary between algebra and geometry, we already see that something is amiss: different ideals can yield the same algebraic subset. Thus the algebraic side carries more information than the geometric side in classical algebraic geometry. This might seem like a minor inconvenience at first, but the deeper one digs into the structure of algebraic varieties, the more essential it becomes. The interpretation of this phenomenon in terms of nilpotents in the structure sheaf of algebraic varieties—and in general the realization of the key role played by nilpotent elements in algebraic geometry—is one of the leading motivations for the abstract language of schemes.*

1.1.1 Examples

At this stage, we cannot say much more, but let us list some basic examples:

1. **Linear varieties.** These are the zero sets of degree one polynomials. They can always be written as $P + W$, where $P \in K^n$ is a point and $W \subset K^n$ is a vector subspace.

2. **Conics in $\mathbb{A}^2_K$.** Assume that $\mathrm{char}(K) \neq 2$ and let $Q(x, y) \in K[x, y]$ be a quadratic polynomial. Then $C = V(Q) \subset K^2$ is called an affine conic. When is C irreducible? If C is not irreducible, in which ways can it decompose?

3. **The Fermat curve.** For $n \geq 1$ let

   $$F_n = V(x^n + y^n - 1) \subset K^2.$$

   Show that F_n is always irreducible (what happens when $\mathrm{char}(K)$ divides n?). Fermat's Last Theorem then says that the only $(a, b) \in \mathbb{Q}^2$ satisfying $a^n + b^n = 1$ are $(1, 0)$ and $(0, 1)$. In fact, much of algebraic number theory boils down to the study of polynomial equations over non-closed fields. Since modern algebraic geometry works with any scheme as a base (not necessarily a field), its language automatically incorporates many classical concepts of algebraic number theory (e.g. ideal class groups, ramification theory, etc.). It will eventually allow one to show how the topological properties of a curve $C \subset \mathbb{C}^2$ (or, more generally, of an algebraic variety) influence the set $C \cap \mathbb{Q}^2$, assuming that C can be defined by equations with coefficients in $\mathbb{Q}$.

4. **Hypersurfaces.** A hypersurface is by definition $V(f) \subset K^n$ where $f \in K[x_1, \dots, x_n]$ is an irreducible polynomial. These are heavily studied for many reasons: they are easy to define, yet they already give very rich families of examples (in fact, as we shall see, every variety is birational to some hypersurface).

1.1.2 Localization and open subsets

Let us now recall the operation of localization and how to describe the Zariski open subsets of a variety $X \subset K^n$. Let R be any commutative ring and let $S \subset R$ be a multiplicative subset. We defined the localization $S^{-1}R$ last year: this is a ring together with a natural map

$$\phi : R \to S^{-1}R$$

which is universal among ring maps $f : R \to R'$ such that $f(S) \subset (R')^{\times}$, i.e. every $f(s)$ is invertible for $s \in S$. We have shown that the induced map

$$\operatorname{Spec}(S^{-1}R) \longrightarrow \operatorname{Spec}(R)$$

is an open immersion with image

$$\{\mathfrak{p} \in \operatorname{Spec}(R) : S \cap \mathfrak{p} = \emptyset\}.$$

The two most important multiplicative sets in algebraic geometry are:

- $S = \{1, f, f^2, \dots\}$ for some $f \in R$, in which case we write $S^{-1}R = R_{(f)}$;
- $S = R \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, in which case we write $S^{-1}R = R_{\mathfrak{p}}$.

Recall that if R is a domain, then $S^{-1}R \subset \operatorname{Frac}(R)$ via the universal property of localization. Concerning $R_{(f)}$, we have a natural identification

$$R_{(f)} \cong \left\{ \frac{h}{f^n} : h \in R,\ n \geq 0 \right\} \subset \operatorname{Frac}(R).$$

Now let $A = K[x_1, \dots, x_n]/\mathfrak{p}$ and let $X = V(\mathfrak{p}) \subset K^n$ be the associated algebraic variety. For any $f \in A$ consider the open subset $D_f = X \setminus V(f)$. These are called *principal open subsets.*

Lemma 1.8. *The following hold:*

1. *The open sets D_f generate the Zariski topology of X;*
2. *Every D_f has a natural structure of an affine algebraic variety.*

Proof. Point (1) is an exercise. For (2), write $\mathfrak{p} = (f_1, \dots, f_k) \subset K[x_1, \dots, x_n]$, so that

$$R_{(f)} \cong K[x_1, \dots, x_n, y]/(f_1, \dots, f_k, yf - 1).$$

Thus $R_{(f)}$ is itself a finitely generated K-algebra, and hence defines an affine algebraic variety in K^{n+1}. Its points correspond uniquely to maximal ideals of $R_{(f)}$, and hence to points of D_f. $\square$

In general, if $X \subset K^n$ is an algebraic variety and $U \subset X$ is an open subset, we say that $U \subset K^n$ is a quasi affine variety.

Excercise 1.9. Show that $\mathrm{GL}_n(K) \subset M_n(K)$ has a natural structure of an affine algebraic variety, where we identify $M_n(K) \cong K^{n^2}$.

1.2 Dimension theory

Recall the definition of the Krull dimension of a ring R: it is the maximal length of a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

(if such a maximum exists). Similarly, for a topological space X, we define $\dim(X)$ as the maximal length of a chain of non-empty irreducible closed subsets

$$Z_n \subsetneq \cdots \subsetneq Z_1 \subsetneq Z_0.$$

So if $A = K[x_1, \cdots, x_n]/I$ is a finitely generated K-algebra $\dim(A) = \dim(V(I))$. We also proved that if A is an integral finitely generated K-algebra then

$$\dim(A) = \mathrm{tr.deg.}_K(\mathrm{Frac}(A)).$$

One would like to have a well-defined notion of codimension. Clearly we could just define $\mathrm{codim}(Z) = \dim(X) \setminus \dim(Z)$ where $Z \subset X$ is a closed subset, but this is an ad-hoc definition which is not well-behaved in practice. The only thing we can do algebraically is the following:

Definition 1.10. Let $\mathfrak{p} \subset R$ be a prime. Then its height $\mathrm{ht}(\mathfrak{p})$ is the maximal length of a chain of prime ideals $\mathfrak{p}_n \subsetneq \cdots \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}$ (if it exists).

The problem now is to show that

$$\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R).$$

But this is not true for general rings, and the right notion to consider is the following:

Definition 1.11. A ring R is *catenary* if for any two primes $\mathfrak{p}_1 \subset \mathfrak{p}_2$, all maximal chains of prime ideals starting at $\mathfrak{p}_1$ and ending at $\mathfrak{p}_2$ have the same finite length.

It is easy to see that any localization or quotient of a catenary ring is still catenary (prove it). The first counterexample of a Noetherian ring which is not catenary was constructed by Nagata (he also constructed the first example of a Noetherian ring with infinite Krull dimension). On the other hand, most if not all of the rings appearing in algebraic geometry are catenary:

Theorem 1.12. *If A is a finitely generated integral K-algebra then it is catenary.*

Proof. The proof can be found in Matsumura's book. We advise the reader to take a look at it. $\square$

Note that without the integrality condition the result clearly fails. For example, consider

$$A = K[x, y, z]/(xy, xz).$$

Then A is not integral (picture it in K^3). One can check that both chains

$$(y, z) \subset (x, y, z), \qquad (x) \subset (x, y) \subset (x, y, z)$$

are maximal chains of prime ideals of different lengths. What is happening geometrically?

Proposition 1.13. *Let R be integral, catenary, and of finite dimension. Then for any prime $\mathfrak{p} \subset R$ we have*

$$\dim(R) = \operatorname{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}).$$

Proof. The proof follows by concatenating chains of prime ideals starting with (0) and ending with $\mathfrak{p}$ with chains of prime ideals in $R/\mathfrak{p}$, and using catenarity to ensure equality of lengths. $\square$

We shall return to the notion of dimension again later in the course.

1.3 Morphisms of varieties

Every time we define new objects in mathematics we also have to define morphisms between them. We have two ways to do so in the case of affine algebraic varieties: let $X \subset K^n$ and $Y \subset K^m$ be two varieties:

1. A map $f : X \to Y$ is a morphism if there are $P_1, \cdots, P_m \in K[x_1, \cdots, x_n]$ such that $f(x) = (P_1(x), \cdots, P_m(x))$ for every $x \in X$.

2. A map $f : X \to Y$ is a morphism if for every $P \in X$ there are polynomial functions $H, G : K^n \to K^m$ such that $G(u) \neq 0$ and $f(u) = H(u)/G(u)$ for every $u \in U \cap X$.

The two definitions turn out to be equivalent, but a priori the first definition is only contained in the second, and it is more rigid as it forces all morphisms to be restrictions of 'global' polynomial maps between K^n and K^m. For example, the first definition does not work for quasi-affine subsets, but the second does. Thinking more about the difference between the two definitions naturally leads to the notion of sheaves. We adopt the first definition for the time being. Finally, we say that X and Y are isomorphic if there are morphisms $f : X \to Y$ and $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.

- Show that any $f \in \mathrm{Hom}(X, Y)$ is continuous for the Zariski topology.

- Since K is a ring $\mathrm{Mor}(X, K)$ is also a ring. Show that $\mathrm{Mor}(X, K) \cong A$ where $X = V(I) \subset K^n$ and $A = K[x_1, \cdots, x_n]/\sqrt{I}$.

- In particular, take $f \in A$. Then $\mathrm{Mor}(D_f, K) \cong R_{(f)}$, where we give D_f the structure of an algebraic variety as before.

- (Linear projections) Let W be a finite dimensional K-vector space. Let $\phi : K^n \to W$ be a linear surjective morphism and let $X \subset K^n$ be an algebraic variety. Then the restriction $\phi_{|X}$ is called a linear projection from X to W.

- (Projections from a point) Choose a point $P \in K^n$ and a linear hyperplane $H \subset K^n$ (that is, the translate of a linear subspace $V \subset K^n$ of dimension $n - 1$ or, equivalently, the zero set of a degree one polynomial $F \in K[x_1, \cdots, x_n]$) such that $P \notin H$. For any point $x \in K^n \setminus P$ let ℓ_x be the line joining P and x. Then only two things can happen: either $\ell_x \cap H$ consists of one point or $\ell_x \cap H = \emptyset$. This second case happens only if $x \in H'$, the unique hyperplane parallel to H with $P \in H'$. So on $D_{H'} = K^n \setminus H'$ we obtain a well-defined set-theoretic map $D_{H'} \to H$, called the projection from P to H. Show that this is a morphism of algebraic varieties. We can restrict this to any algebraic variety and obtain a morphism defined on a principal open subset.

- (Affine conics, continued). Consider an irreducible conic $C = V(Q) \subset K^2$ and assume that $(0, 0) \in C$, which is always possible up to translation. We want to show that either C is isomorphic to K or to $K \setminus 0 = D_x$ as algebraic varieties. In fact, we can do this in two ways. To prove the result algebraically, write $Q = Q_2 + Q_1$ where Q_2 is homogeneous of degree 2 and Q_1 is homogeneous of degree 1 (there is no constant term since $(0, 0) \in Q$). So $Q_2(x, y) = ax^2 + bxy + cx^2$ which is the quadratic form associated to the bilinear form $\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ (recall that $\mathrm{char}(K) \neq 2$). Then Q_2 as a matrix has rank either one or two. With a linear change of variable, show that in the first case C is isomorphic to the parabola $V(y - x^2) \cong K$ and that in the second one to the hyperbole $V(xy - 1) \cong K \setminus 0$.

  One can also prove the result geometrically, but the light on this will be shed once we introduce projective spaces, and see the conic as a curve inside the projective plane. One begins by answering the following: how many lines $L \subset K^2$ passing through $(0, 0)$ are such that $L \cap C = \{(0, 0)\}$?

Clearly the tangent line $T \subset K^2$ is one of them. Show that there are either one or two more, and fix one of them and call it L. Finally, for any point $P' \in C$ let L' be the unique line parallel to L and passing through P'. Show that $L' \cap T$ consists of a unique point $\phi(P')$. Show that this defines a morphism $\phi : C \to T$. Show that in the first case (where there is only one line L) the map is an isomorphism, and in the second, the map is an isomorphism onto the complement of a point.

- (Frobenius) Assume that $\mathrm{char}(K) = p > 0$ and let $X = V(\mathfrak{p}) \subset K^n$ be a variety. Assume for simplicity that $\mathfrak{p} = (f_1, \cdots, f_d)$ where $f_i \in \mathbb{F}_p[x_1, \cdots, x_n] \subset K[x_1, \cdots, x_n]$ for every i. Now consider the map $F : K^n \to K^n$ sending $(a_1, \cdots, a_n) \mapsto (a_1^p, \cdots, a_n^p)$, which corresponds to the K-algebra map $K[x_1, \cdots, x_n] \to K[x_1, \cdots, x_n]$ sending $x_i \mapsto x_i^p$. Show that F induces a bijection $F : X \to X$, which is called the geometric Frobenius of X. Note that we would expect the map $F : X \to X$ to have degree $p^{\dim(X)}$ (for instance, try to show that the induced map of function fields $\mathrm{Frac}(A(X)) \to \mathrm{Frac}(A(X))$ yields a finite extension of degree $p^{\dim(X)}$) but topologically the map is a bijection, hence it should be an isomorphism. This ‘problem’ is also solved by schemes, for which indeed the map turns out to have the right degree.

Now, let $X \subset K^n$ be given by $V(\mathfrak{p})$ and $Y \subset K^m$ be given by $V(\mathfrak{q})$.

Proposition 1.14. *There is a natural identification*

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}_{K\text{-}alg}(A(Y), A(X))$$

where, from now on, we write $A(X) = \mathrm{Hom}(X, K)$.

Proof. Take a morphism $f : X \to Y$. Then the induced map $A(Y) \to A(X)$ is simply given by composition

$$A(Y) \ni (g : Y \to K) \mapsto (g \circ f : X \to K) \in A(X),$$

which defines a ring morphism.

For the other direction, write $A(X) = K[x_1, \cdots, x_n]/\mathfrak{p}$ and $A(Y) = K[x_1, \cdots, x_m]/\mathfrak{q}$. Consider a map of rings $\phi : A(Y) \to A(X)$; this induces a map of rings $K[x_1, \cdots, x_m] \to A(X)$ which sends x_i to some $f_i' \in A(X)$. Now $A(X)$ is quotient of $K[x_1, \cdots, x_n]$. Find $f_i \in K[x_1, \cdots, x_n]$ which reduce to $f_i' \in A(X)$; by freeness of $K[x_1, \cdots, x_n]$ we can thus define define a morphism $K[x_1, \cdots, x_m] \to K[x_1, \cdots, x_n]$ hence a morphism $f : K^n \to K^m$ sending $(k_1, \cdots, k_n) \mapsto (f_1(k_1, \cdots, k_n), \cdots, f_m(k_1, \cdots, k_n))$.

The facet that $f(X) \subset Y$ then follows from the commutativity of

$$\begin{tikzcd}
K[x_1, \cdots, x_m] \arrow[r] \arrow[d, two heads] & K[x_1, \cdots, x_n] \arrow[d, two heads] \\
A(Y) \arrow[r] & A(X)
\end{tikzcd}$$

These constructions are one the inverse of the other. □

2 Lecture II

2.1 Presheaves

The philosophy behind sheaves comes precisely from the two different possible definitions of morphisms between algebraic varieties. Although the concept of sheaf is not always stated explicitly, it is already present in many constructions in analysis. For example, let $U \subset \mathbb{C}^n$ be an open subset (with the usual topology) and let $f : U \to \mathbb{C}$ be a continuous function. Then f is said to be holomorphic on U if, for every point $P \in U$, there exists an open neighborhood $V \subset U$ of P over which f can be expressed as a converging power series. In other words, a holomorphic function is defined by its *local behaviour*, and typically we cannot express f globally on all of U as a single converging power series. This illustrates the principle that global objects are determined by compatible local data, which is exactly what the concept of a sheaf formalizes.

Definition 2.1. Let X be a topological space. A *presheaf* of abelian groups $\mathscr{F}$ on X consists of the following data:

1. For every open subset $U \subset X$, an abelian group $\mathscr{F}(U)$;
2. For every inclusion of open sets $V \subset U$, a *restriction map*

$$\rho_{U,V} : \mathscr{F}(U) \longrightarrow \mathscr{F}(V)$$

   satisfying the following:

   (a) $\rho_{U,U} = \mathrm{Id}_{\mathscr{F}(U)}$;
   (b) For any chain $U_1 \subset U_2 \subset U_3$, we have $\rho_{U_2,U_1} \circ \rho_{U_3,U_2} = \rho_{U_3,U_1}$.

An element $s \in \mathscr{F}(U)$ is called a *section* of $\mathscr{F}$ over U, and we write

$$s|_V := \rho_{U,V}(s)$$

for its restriction to $V \subset U$. The group $\mathscr{F}(X)$ is often denoted $\Gamma(X, \mathscr{F})$ and its elements are called *global sections*.

Remark 2.2. Similarly, one can define presheaves of sets, rings, or other objects. In fact, for any category $\mathscr{C}$, a presheaf with values in $\mathscr{C}$ can be defined in the same way. Equivalently, let $\mathrm{Open}(X)$ be the category whose objects are open subsets of X, with

$$\mathrm{Hom}(U, V) = \begin{cases} \{*\}, & \text{if } U \subset V, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then a presheaf with values in $\mathscr{C}$ is exactly a contravariant functor

$$\mathscr{F} : \operatorname{Open}(X) \longrightarrow \mathscr{C},$$

where $\mathscr{F}(U)$ is the value on U and the restrictions are given by the functorial action.

2.1.1 Examples

We now make some important examples of presheaves, divided into two classes.

1. Structure sheaves of geometric objects.

   All the following are presheaves of rings, with restriction maps given by restricting functions to smaller open sets.

   (a) *Continuous functions:* Let X be a topological space. Define

   $$\mathscr{C}_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is continuous}\}.$$

   (b) *Differentiable functions:* Let X be a $\mathscr{C}^\infty$-manifold. Define

   $$\mathscr{C}_X^\infty(U) = \{f : U \to \mathbb{R} \mid f \text{ is smooth}\}.$$

   (c) *Holomorphic functions:* Let X be a complex manifold. Define

   $$\mathscr{O}_X^{\text{hol}}(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}.$$

   (d) *Regular functions:* Let $X \subset K^n$ be an algebraic variety with the Zariski topology. Define

   $$\mathscr{O}_X(U) = \left\{ f : U \to K \;\middle|\; \begin{array}{c} \text{for every } P{\in}U \text{ there is an open } P{\in}V{\subset}U \text{ and} \\ \text{polynomials } h,g{\in}K[x_1,\cdots,x_n] \text{ such that} \\ g(v){\neq}0 \text{ and } f(v){=}h(v)/g(v) \text{ for every } v{\in}V \end{array} \right\}.$$

2. Constant presheaves.

   (a) Let G be an abelian group. The constant presheaf with values in G is

   $$\underset{\sim}{G}(U) = \{f : U \to G \mid f \text{ is constant}\}.$$

   (b) Endow G with the discrete topology. The *locally constant presheaf* $\underline{G}$ is defined by

   $$\underline{G}(U) = \{f : U \to G \mid f \text{ is continuous}\}.$$

(c) Let $\pi : Y \to X$ be a covering map of topological spaces. Define the presheaf of sections

$$\mathcal{S}_\pi(U) = \{f : U \to Y \mid f \text{ continuous and } \pi \circ f = \mathrm{Id}_U\}.$$

We now discuss some differences and similarities between the examples of presheaves introduced before. The first and most important observation is that all the presheaves in point (1) satisfy some extra fundamental properties.

Let $\mathcal{O}$ be any of the presheaves from point (1), let $U \subset X$ be an open subset, and let $\{U_i\}_{i\in I}$ be any open cover of U. Then the following hold:

I. **(Gluing of functions)** Given sections $f_i \in \mathcal{O}(U_i)$ such that

$$f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j} \quad \text{for all } i, j \in I,$$

there exists a section $f \in \mathcal{O}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

II. **(Local nature of functions)** If $f, g \in \mathcal{O}(U)$ are such that

$$f|_{U_i} = g|_{U_i} \quad \text{for all } i \in I,$$

then $f = g$.

These two properties are exactly what distinguish *sheaves* from general presheaves. Let us see how these properties can fail:

Failure of gluing (I). Consider a topological space $X = X_1 \sqcup X_2$ with $X_1, X_2 \neq \emptyset$, and let $\tilde{G}$ be the constant presheaf defined earlier. Then

$$\tilde{G}(X_1) = G, \quad \tilde{G}(X_2) = G, \quad \tilde{G}(\emptyset) = 0.$$

Any two sections $g_1 \in \tilde{G}(X_1)$ and $g_2 \in \tilde{G}(X_2)$ satisfy the local compatibility condition, but if $g_1 \neq g_2$, they cannot be glued to a global section in $\Gamma(X, \tilde{G})$. In contrast, the locally constant presheaf $\underline{G}$ satisfies the gluing property and we have $\underline{G}(X) = G^{\pi_0(X)}$ (at lease when all connected components of X are open).

Failure of locality (II). To construct a presheaf that fails property (II) is slightly more subtle. Consider the presheaf of bounded continuous functions

$$\mathcal{C}^b_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is bounded}\} \subset \mathcal{C}_X(U).$$

Then the quotient presheaf

$$\mathcal{F}(U) = \mathcal{C}_X(U)/\mathcal{C}^b_X(U)$$

does not satisfy (II). For example, if $X = \mathbb{R}$, consider the section represented by $x \in \mathcal{C}_X(\mathbb{R})$. Then $0 \neq x \in \mathcal{F}(\mathbb{R})$, but for any open cover $\{U_i\}_{i\in I}$ of $\mathbb{R}$ such that each $\overline{U_i}$ is compact we have $x|_{U_i} = 0$ in $\mathcal{F}(U_i)$.

2.2 Sheaves

Definition 2.3. A presheaf $\mathscr{F}$ is called a *sheaf* if for every open subset $U \subset X$ and every open cover $\{U_i\}_{i \in I}$ of U, the following hold:

I. **(Gluing of sections)** Given sections $s_i \in \mathscr{F}(U_i)$ such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \text{for all } i, j \in I,$$

there exists a section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

II. **(Locality)** If $s, s' \in \mathscr{F}(U)$ satisfy

$$s|_{U_i} = s'|_{U_i} \quad \text{for all } i \in I,$$

then $s = s'$.

Remark 2.4. Property (II) guarantees that the gluing in (I) is unique.

Now, there is a canonical way to obtain a sheaf from a presheaf but first we need talk about stalks (note: the terminology sheaves and stalks is not random). Stalks are used to study sections around a specific point $P \in X$:

Definition 2.5. Let $\mathscr{F}$ be a presheaf and let $P \in X$. The *stalk* of $\mathscr{F}$ at P is

$$\mathscr{F}_P = \varinjlim_{P \in U} \mathscr{F}(U),$$

where the colimit runs over all open neighborhoods U of P. An element of $\mathscr{F}_P$ is called a *germ* of a section at P. Similarly, for a subset $Z \subset X$, the stalk at Z is defined by

$$\mathscr{F}_Z = \varinjlim_{Z \subset U} \mathscr{F}(U).$$

Remark 2.6. In practice, a germ in $\mathscr{F}_P$ is represented by a pair (U, s) with $P \in U$ and $s \in \mathscr{F}(U)$. Two such pairs (U, s) and (U', s') define the same germ if there exists an open $V \subset U \cap U'$ containing P such that $s|_V = s'|_V$. If $\mathscr{F}$ takes values in a category $\mathscr{C}$, we assume that $\mathscr{C}$ admits direct limits so that this definition makes sense.

Remark 2.7. For the sheaves of functions introduced earlier, the restriction maps $\rho_{U,V}$ have special properties: they are surjective for $\mathscr{C}_X$ and $\mathscr{C}_X^\infty$ (due to the existence of partitions of unity), and injective for $\mathscr{O}_X^{\text{hol}}$ and $\mathscr{O}_X$ (because holomorphic and regular functions are rigid: if they agree on an open subset, they agree everywhere).

Let now $X = V(\mathfrak{p}) \subset K^n$ be a variety, and consider the sheaf $\mathcal{O}_X$ more closely. Let $K(X) = \mathrm{Frac}(A(X))$.

Lemma 2.8. *For any open subset $U \subset X$, there is a natural inclusion*

$$\mathcal{O}_X(U) \subset K(X).$$

Proof. Let $f \in \mathcal{O}_X(U)$. By definition, there exists a finite open cover U_i of U and polynomials $P_i, Q_i \in K[x_1, \dots, x_n]$ such that $Q_i(u) \neq 0$ for all $u \in U_i$ and $f(u) = P_i(u)/Q_i(u)$ on U_i. Now $Q_i \notin \mathfrak{p}$ for otherwise $Q_i(x) = 0$ for every $x \in X$. Hence Q_i is invertible in the localization $K[x_1, \cdots, x_n]_{\mathfrak{p}}$ and so it defines an element $[Q_i] \in K(X) = K[x_1, \cdots, x_n]_{\mathfrak{p}}/\mathfrak{p}$. We claim that $[P_i][Q_i]^{-1} \in K(X)$ is well-defined and independent of i. First, if $P_i(u)/Q_i(u) = P_i'(u)/Q_i'(u)$ for every $u \in U_i$ then $P_iQ_i' - P_i'Q_i$ defines an element of $A(X)$ whose vanishing locus contains U_i, which is a non-trivial Zariski open. If $P_iQ_i' - P_i'Q_i$ was non-zero in $A(X)$ then also its non-vanishing locus would be a non-empty Zariski open. Since any two Zariski opens intersect because X is irreducible, we see that $P_iQ_i' - P_i'Q_i \in \mathfrak{p}$ and thus $[P_i][Q_i]^{-1} \in K(X)$ does not depend on the representatives P_i, Q_i over U_i. Similarly (i.e., using the fact that $U_i \cap U_j \neq \emptyset$) one shows that $[P_i][Q_i]^{-1} = [P_j][Q_j]^{-1}$ as well.

This defines a map $\mathcal{O}_X(U) \to K(X)$. Suppose that this were not injective. Then $[P_i] \in A(X)$ must vanish on a non-trivial Zariski open - hence it vanishes everywhere - and therefore $f = 0$. □

Remark 2.9. This also shows that the restriction maps $\rho_{U,V}$ for $\mathcal{O}_X$ are injective whenever $V \neq \emptyset$.

Proposition 2.10. *Let $X = V(\mathfrak{p}) \subset K^n$ be as before.*

1. *For every $x \in X$ the stalk $\mathcal{O}_{X,x}$ is isomorphic to the localization $A(X)_{\mathfrak{m}_x}$.*
2. *We can identify*
$$\mathcal{O}_X(U) = \bigcap_{x \in U} A(X)_{\mathfrak{m}_x} \subset K(X)$$
3. *In particular, $\Gamma(X, \mathcal{O}_X) = A(X)$ (as promised).*

Proof.

- Using the previous lemma we have
$$\mathcal{O}_{X,x} = \{f/g : f, g \in A(X) \text{ and } g(x) \neq 0\} \subset K(X).$$
- The inclusion $\mathcal{O}_X(U) \subset \bigcap_{x \in U} A(X)_{\mathfrak{m}_x}$ is clear. Let now h be in the intersection, then for every $x \in U$ we can write $f = h_x/g_x$ for some

$f_x, g_x \in A(X)$ such that $g_x(x) \neq 0$. So g_x does not vanish in a neighbrhood $x \in U_x \subset X$ of x. But then there are finitely many $x_1, \cdots, x_n$ such that $U_i = U_{x_i}$ cover X and such that $g_i(u) \neq 0$ for every $u \in U_i$. Thus, the function $U \to K$ sending $u \mapsto f_i(u)/g_i(u)$ if $u \in U_i$ is well-defined and defines a section of $\mathcal{O}_X(U)$.

- Let $h \in \bigcap_{x \in X} A(X)_{\mathfrak{m}_x}$ and consider the ideal $I = \{g \in A(X) : gh \in A(X)\} \subset A(X)$. If this is not trivial, then it is contained in a maximal ideal $\mathfrak{m} \subset A(X)$ corresponding to some $x \in X$. But $h \in A(X)_{\mathfrak{m}}$ so $h = f/g$ with $f \in A(X)$ and $g(x) \neq 0$ and hence also $g \in I$, contradicting that $I \subset \mathfrak{m}$.

□

The fact that a sheaf is determined by its local behavious is captured in a number of propositions, for example:

Proposition 2.11. *If $\mathcal{F}$ is a sheaf then for every open $U \subset X$ the natural map*

$$\mathcal{F}(U) \to \prod_{P \in U} \mathcal{F}_P$$

is injective.

Proof. If a $s \in \mathcal{F}(U)$ is in the kernel of the above, then by the definition of stalks, we can find for every point P an open $P \in U_P$ such that $s_{|U_P} = 0$. Since $\{U_P\}_P$ covers X the result follows by condition (II). □

Note that in fact one can also characterize the image:

Proposition 2.12. *If $\mathcal{F}$ is a sheaf then the image of*

$$\mathcal{F}(U) \to \prod_{P \in U} \mathcal{F}_P$$

corresponds to

$$\left\{ (s_P) \in \prod_{P \in U} \mathcal{F}_P : \begin{array}{c} \textit{for every } P \textit{ there is } U \subset X \textit{ open with} \\ P \in U \textit{ and a section } S \in \mathcal{F}(U) \textit{ such} \\ \textit{that } s_Q = S_Q \textit{ in } \mathcal{F}_Q \textit{ for every } Q \in U \end{array} \right\}.$$

Proof. Use the previous proposition and the glueing condition. □

Sheafification So now it is easy to associate a sheaf to any presheaf

Definition 2.13. If $\mathscr{F}$ is a presheaf its sheafification $\mathscr{F}^+$ is defined by

$$\mathscr{F}^+(U) := \left\{ (s_P) \in \prod_{P\in U} \mathscr{F}_P : \begin{array}{c} \text{for every } P \text{ there is } U{\subset}X \text{ open with} \\ P{\in}U \text{ and a section } S{\in}\mathscr{F}(U) \text{ such} \\ \text{that } s_Q{=}S_Q \text{ in } \mathscr{F}_Q \text{ for every } Q{\in}U \end{array} \right\}.$$

This is a sheaf and there is a natural map of presheaves $\mathscr{F} \to \mathscr{F}^+$ which is an *isomorphism* on stalks. It satisfies a universal property: for any sheaf $\mathscr{G}$ any morphism $\mathscr{F} \to \mathscr{G}$ factorises uniquely as $\mathscr{F} \to \mathscr{F}^+ \to \mathscr{G}$.

Remark 2.14. To get another glimpse into the local nature of sheaves, consider the following example: let $\mathscr{P}$ be a property on the open subsets of X such that if $U \subset V$ and $\mathscr{P}(V)$ is true then also $\mathscr{P}(U)$ is true. Define the presheaf of sets $\mathscr{F}(U) = \{*\}$ if $\mathscr{P}(U)$ is true and $\mathscr{F}(U) = \emptyset$ if $\mathscr{P}(U)$ is false and let $\mathscr{F}^+$ be its sheafification. Then $\Gamma(X, \mathscr{F}) = \{*\}$ if and only if $\mathscr{P}(X)$ is true whereas $\Gamma(X, \mathscr{F}^+) = \{*\}$ if and only if $\mathscr{P}$ is *locally true* over X, that is, X can be covered by open subsets U_i such that $\mathscr{P}(U_i)$ is true for every i.

2.3 Exact properties of sheaves and presheaves

Definition 2.15. Let $\mathscr{F}, \mathscr{G}$ be presheaves on X of sets, abelian groups, etc. A morphism $\phi : \mathscr{F} \to \mathscr{G}$ is by definition a collection of maps of sets, abelian groups, etc $\phi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for every open $U \subset X$ satisfying the obvious compatibility with the restriction maps.

1. ϕ is injective if ϕ_U is injective for every U;
2. ϕ is surjective if ϕ_U is surjective for every U.

Assume now that both $\mathscr{F}$ and $\mathscr{G}$ are presheaves of abelian groups:

1. $\ker(\phi)$ is the presheaf $U \mapsto \ker(\phi_U)$;
2. $\operatorname{coker}(\phi)$ is the presheaf $U \mapsto \operatorname{coker}(\phi_U)$
3. $\operatorname{Im}(\phi)$ is the presheaf $U \mapsto \operatorname{Im}(\phi_U)$.

Remark 2.16. One can form a category of presheaves of abelian groups on X which turns out to be an abelian category. The definitions above then corresponds to the abstract definitions of the same objects in any abelian category.

Remark 2.17. Recall that direct limits preserve exactness. In particular, if $\phi : \mathscr{F} \to \mathscr{G}$ is injective/surjective then the induced map $\phi_P : \mathscr{F}_P \to \mathscr{G}_P$ is injective/surjective for every point P.

For sheaves, the situation is different. One can again form the category of sheaves of abelian groups on X, which again turns out to be an abelian category. But it might be that $\mathscr{F} \to \mathscr{G}$ is an epimorphism of sheaves without being an epimorphism of presheaves:

Proposition 2.18 (Exactness properties of sheaves). *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves. Then:*

1. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves also the presheaf* $\ker(\phi)$ *is a sheaf;*
2. *If ϕ_U is injective for every open U then also $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is injective;*
3. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves then the sheafification of* $\mathrm{Im}(\phi)$ *is naturally a subsheaf of $\mathscr{G}$;*
4. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves then ϕ is an epimorphism of sheaves if and only if ϕ_P is surjective for every P.*

Proof.

1. Clearly $\ker(\phi)$ satisfies (II), so let us show that we can glue sections. Let $\bigcup_i U_i = U$ be an open cover of some open $U \subset X$ and let $s_i \in \ker(\phi_{U_i})$ be local sections. Assume that they satisfy the glueing condition. Then, since $\mathscr{F}$ is a sheaf we can at least find some $s \in \mathscr{F}(U)$ such that $s_{|U_i} = s_i$ where equality is taken inside $\mathscr{F}(U_i)$. So consider $\phi(s) \in \mathscr{G}(U)$. But $\phi(s)_{|U_i} = \phi(s_{|U_i}) = \phi(s_i) = 0$. Since $\mathscr{G}$ is a sheaf we then have $\phi(s) = 0$, i.e., $s \in \ker(\phi_U)$.
2. By construction of the sheafification, it is sufficient to show that the induced map of stalks $\phi_P : \mathscr{F}_P \to \mathscr{G}_P$ is injective for every P, which follows by the previous remark.
3. Since the map of presheaves $\mathrm{Im}(\phi) \to \mathscr{G}$ is injective by definition also $\mathrm{Im}(\phi)^+ \to \mathscr{G}^+ = \mathscr{G}$ is injective by the previous point.
4. In fact, ϕ is an epimorphism if and only if $\mathrm{Im}(\phi)^+ = \mathscr{G}$. By what we have shown, this is equivalent to $\mathrm{Im}(\phi)^+_P = \mathscr{G}_P$ for every point P. But $\mathrm{Im}(\phi)^+_P = \mathrm{Im}(\phi_P) = \mathscr{G}_P$ by assumption.

□

Now, in any abelian category, if a morphism is both an epimorphism (surjective) and a monomorphism (injective) it must be invertible. Let us verify this:

Proposition 2.19. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphim of sheaves (not necessarily of abelian groups) such that ϕ_P is an isomorphism for every P. Then ϕ is an isomorphism.*

Proof. We know that $\phi_{|U}$ is injective for every $U \subset X$ open. We only need to show that it is also surjective. So pick $s \in \mathscr{G}(U)$ and let $P \in U$. Then we can find by assumption some $t_P \in \mathscr{F}_P$ such that $\phi_P(t_P) = s_P$. Let t_P be represented by $t'_P \in \mathscr{F}(V'_P)$ where $P \in V'_P \subset U$ is some open subset. Then the various V'_P cover X. Moreover, the various t'_P agree on the overlappings $V'_P \cap V'_Q$ due to the injectivity of ϕ, implying that they satisfy the glueing condition - hence the result. $\square$

Returning to our discussion, the fact that surjectivity for sheaves is a local condition naturally leads to sheaves cohomology: consider a short exact sequence of abelian sheaves

$$0 \to \mathscr{F}_1 \xrightarrow{f} \mathscr{F}_2 \xrightarrow{g} \mathscr{F}_3 \to 0$$

meaning that $\mathscr{F}_1 = \ker(g)$ and $\mathscr{F}_3 = \operatorname{coker}(f)$. Then

$$0 \to \Gamma(X, \mathscr{F}_1) \xrightarrow{f_X} \Gamma(X, \mathscr{F}_2) \xrightarrow{g_X} \Gamma(X, \mathscr{F}_3)$$

is exact, but g_X need not be surjective. So $\Gamma(X, -)$ is a left-exact functor and one we study its derived functors, whose role in this case is to measure the obstructions that prevent local data from glueing into global sections. From the behaviour of such cohomology groups we then get information about the geometry of X. This will be done in the second part of the course. Let us see now a nice example which mixes complex analysis and topology:

Example 2.20. Let $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}^\times \subset \mathbb{C}$ be the circle group with its usual topology. Let $\mathscr{C}_{\mathbb{S},\mathbb{C}}$ be the sheaf of continuous functions from $\mathbb{S}$ to $\mathbb{C}$ and let $\mathscr{C}^\times_{\mathbb{S},\mathbb{C}}$ be the sheaf of continuous invertible functions from $\mathbb{S}$ to $\mathbb{C}^\times$. Then we have a short exact sequence of sheaves (verify it)

$$0 \to (2\pi i)\underline{\mathbb{Z}} \to \mathscr{C}_{\mathbb{S},\mathbb{C}} \xrightarrow{\exp} \mathscr{C}^\times_{\mathbb{S},\mathbb{C}} \to 0$$

where $\underline{\mathbb{Z}}$ is the locally constant sheaf and exp sends a local function $g : U \to \mathbb{C}$ to $\exp(g)$, where $U \subset \mathbb{S}$. Let now $f : \mathbb{S} \to \mathbb{C}^\times$ be a global section $f \in \Gamma(\mathbb{S}, \mathscr{C}^\times_{X,\mathbb{C}})$. By fixing $1 \in \mathbb{S}$ we can consider f as a continuous loop in $\mathbb{C}^\times$ with base point $f(1)$, and so f yields an element $[f] \in \pi_1(\mathbb{C}^\times, f(1)) \cong \mathbb{Z}$. Show that $f = \exp(g)$ for some global section $g : \mathbb{S} \to \mathbb{C}$ if and only if $[f] = 0$.

Definition 2.21 (Funcoriality of sheaves). Let X, Y be topological spaces, $\mathscr{F}$ a sheaf on X, $\mathscr{G}$ a sheaf on Y and $f : X \to Y$ a continuous map.

1. Push-forward (or direct image) One defines the sheaf $f_*\mathscr{F}$ on Y by the rule $U \mapsto \mathscr{F}(f^{-1}(U))$. Show that this is a sheaf.

2. Inverse image One defines the sheaf $f^{-1}\mathscr{G}$ on X as the sheafification of $\overline{U \mapsto \mathscr{G}_{f(U)}}$ (the stalk of $\mathscr{G}$ at $f(U)$).

Although f_* is easier to define, it is easier to compute the stalks of f^{-1}:

Proposition 2.22. *For $x \in X$ we have natural isomorphisms:*

1. $(f_*\mathscr{F})_{f(x)} \cong \mathscr{F}_{f^{-1}(f(x))}$.
2. $(f^{-1}\mathscr{G})_x = \mathscr{G}_{f(x)}$.

Proof. This follows directly from the definition. □

In particular, f^{-1} is exact and f_* is left-exact. We shall define f^* later in the contenxt of schemes. Finally, suppose we are in the situation of the previous definition, and that we want to define a morphism of sheaves $\mathscr{G} \to \mathscr{F}$ 'along f'. We can then either consider $\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$ or $\mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$: the two are the same.

Proposition 2.23 (adjunction inverse/direct image). *There is a natural isomorphism* $\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$.